

Deformations of associative submanifolds with boundary

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Abstract

Let M be a compact smooth manifold of holonomy G_2 . We prove that the space of infinitesimal associative deformations of a compact associative submanifold Y with boundary in a coassociative submanifold X is the solution space of an elliptic problem. Further, we compute its virtual dimension. For ∂Y connected it is given by $\int_{\partial Y} c_1(\nu_X) + 1 - g$, where g denotes the genus of ∂Y , ν_X the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to its natural complex structure.

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1 Introduction

The group G_2 is one of the possible holonomy groups of an irreducible and non-symmetric Riemannian manifold. As such, manifolds of holonomy G_2 were an active area of research in Riemannian geometry, culminating with Joyce's celebrated construction of compact holonomy G_2 manifolds [18]. As they are necessarily seven dimensional, one refers to G_2 as an *exceptional* holonomy group. In recent years physicists also paid accrued interest to these since the arrival of M theory.

The deep and rich interplay between geometry and algebra on manifolds with a G_2 structure is reflected in the existence of special submanifolds, namely *associative* ones of dimension 3 and *coassociative* ones of dimension 4. These are particular instances of Harvey's and Lawson's *calibrated submanifolds* [13], a notion which also embraces complex submanifolds of a Kähler manifold or special Lagrangian submanifolds of a Calabi–Yau. McLean [25] proved that the infinitesimal coassociative deformations of a coassociative X is an unobstructed elliptic problem. The dimension of the moduli space is $b_+^2(X)$, i.e. the dimension of self-dual harmonic 2-forms on X . For associative submanifolds, the problem, though still elliptic, is more involved: the virtual dimension vanishes, and as for complex submanifolds, existence of deformations is in general obstructed. The work of Akbulut and Salur [2], [3] studies associative deformations on manifolds with topological G_2 structure, but whose holonomy is not necessarily contained in G_2 , and also addresses smooth- and compactness issues of the deformation spaces.

On the other hand, on symplectic manifolds one is naturally led to study the moduli space of (pseudo-)holomorphic curves with boundary in a lagrangian submanifold [11], [12]. In

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physics, Aganagic and Vafa translated this boundary problem for special lagrangians of a Calabi–Yau into an open string problem [1], following Witten’s use of the moduli space of complex curves in the stringy world [26]. Taking a Calabi–Yau 3–fold K times a circle yields a natural holonomy G_2 manifold $M = K \times S^1$. Moreover, holomorphic curves and special lagrangians times a circle give examples of associative and coassociative submanifolds in M . In this way, the duality of complex versus special lagrangian submanifolds is matched by the duality of associative versus coassociative submanifolds in a holonomy G_2 manifold. It is therefore natural to study deformations of (co–)associatives with boundary. Inspired by the work of Butscher [6], who investigated deformations of special lagrangians with boundary on a symplectic, codimension 2 submanifold inside some compact Calabi–Yau, Kovalev and Lotay investigated in a recent paper the analogous problem for manifolds with G_2 structure, where a compact coassociative has its boundary in a fixed, codimension 2 submanifold [21]. On the other hand, Leung and Wang studied associatives with boundary conditions. They consider Riemann surfaces in a fixed coassociative X which they thicken into an associative with boundary in $X \cup X'$, where the coassociative X' is an infinitesimal displacement of X [23].

In this paper, we consider an associative Y with boundary in a fixed coassociative X , and study the space $\mathcal{M}_{X,Y}$ of infinitesimal associative deformations of Y such that the boundary stays in X . Our main result is this: We show that $\mathcal{M}_{X,Y}$ is the solution space of an elliptic boundary value problem whose index (i.e. the virtual dimension of $\mathcal{M}_{X,Y}$) for connected ∂Y is given by

$$\text{index} = \int_{\partial Y} c_1(\nu_X) + 1 - g,$$

where g is the genus of ∂Y , ν_X the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to its natural complex structure. If the boundary is not connected, then the index is the sum over all components of ∂Y . Further, we extend this result to encompass 4–dimensional ϕ –free submanifolds X (i.e. X does not contain any associative) and manifolds with topological G_2 structure, but not necessarily of holonomy G_2 . Finally and independently, assuming that Y is an embedded 3–disk, we associate with Y an element $\mu_{G_2}(\partial Y) \in \pi_2(G_2/SO(4)) \cong \mathbb{Z}_2$, which is best thought of as a G_2 analogon of the Maslov index. Under suitable identifications, we show that

$$\mu_{G_2}(\partial Y) = \int_{\partial Y} c_1(\nu_X) \bmod 2.$$

A further natural issue is to study smooth– and compactness of $\mathcal{M}_{X,Y}$ in the vein of [3], but we will leave this to another paper. The techniques we use are the standard ones from PDE theory; our reference is [4] whose conventions we shall follow throughout this paper.

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2 The group G_2

We start by recalling some classical facts about G_2 (cf. for instance [3], [13] and [19]).

2.1 The octonions

The octonions define an 8-dimensional, non-associative division algebra $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$ generated by $\langle \mathbf{1}, i, j, k, e, e \cdot i, e \cdot j, e \cdot k \rangle$. Taking these generators as an orthonormal basis induces an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{O} compatible with the algebra structure. Further, we obtain a *cross product* taking values in the imaginary octonions $\text{Im } \mathbb{O} = \langle \mathbf{1} \rangle^\perp \cong \mathbb{R}^7$ defined by

$$u \times v = \text{Im}(\bar{v} \cdot u).$$

Here, \bar{v} is the natural conjugation which sends $v \in \text{Im } \mathbb{O}$ to $-v$. The term cross product is justified by the properties $u \times v = -v \times u$ and $|u \times v| = |u \wedge v|$. Over \mathbb{R}^7 , this yields the 3-form

$$\varphi_0(u, v, w) = \langle u \times v, w \rangle, \quad (1)$$

which expressed in the orthonormal basis $e_1 = i, e_2 = k, \dots, e_7 = e \cdot k$ can be written explicitly as

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}). \quad (2)$$

By definition the stabiliser of φ_0 inside $GL(7)$ is G_2 , which is why we refer to any basis $\{e_j\}$ such that φ_0 is of the form (2) as a G_2 *frame*. This is a real algebraic Lie group of dimension 14 defined by the equations

$$G_2 = \{(u_1, u_2, u_3) \in \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \mid \langle u_i, u_j \rangle = \delta_{ij}, \varphi_0(u_1, u_2, u_3) = 0\}. \quad (3)$$

Conversely, any G_2 invariant form $\varphi \in \Lambda^3 \mathbb{R}^{7*}$ induces a positive definite inner product $\langle \cdot, \cdot \rangle_\varphi$ and a cross product \times_φ as follows. Firstly, with φ we can associate a volume form μ_φ (which is somehow difficult to write down explicitly, cf. the appendix in [17]). Then we define

$$\langle u, v \rangle_\varphi = ((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi) / 6\mu_\varphi, \quad \langle u \times_\varphi v, w \rangle_\varphi = \varphi(u, v, w). \quad (4)$$

Further, there exists a *triple cross product* on \mathbb{O} ,

$$C(u, v, w) = \frac{1}{2}(u \cdot (\bar{v} \cdot w) - z \cdot (\bar{y} \cdot x)). \quad (5)$$

Again, this operation is skew in its arguments and satisfies $|C(u, v, w)| = |u \wedge v \wedge w|$. Restricted to \mathbb{R}^7 , we find $\text{Re } C(u, v, w) = \varphi_0(u, v, w)$ so that on G_2 ,

$$C(u, v, w) = \text{Im } C(u, v, w) = \frac{1}{2}((u \cdot v) \cdot w - u \cdot (v \cdot w)).$$

The last expression is sometimes written as $[u, v, w]$ and called the *associator*. It is actually a 3-form over \mathbb{O} , and in analogy with (1), we define a 4-form over \mathbb{R}^7 by

$$\psi_0(u, v, w, x) = \frac{1}{2}\langle u, [v, w, x] \rangle.$$

This form actually coincides with the Hodge dual of φ_0 , so that in a G_2 frame $\{e_j\}$,

$$\psi_0 = \star \varphi_0 = -e^{12} \wedge (e^{47} + e^{56}) - e^{13} \wedge (e^{46} - e^{57}) + e^{23} \wedge (e^{45} + e^{67}) + e^{4567}.$$

2.2 Associative and coassociative planes

An oriented 3-plane $Y \subset \mathbb{R}^7$ is *associative* if the 3-form φ_0 restricted to Y coincides with the induced Euclidean volume form on Y . By $G^{\varphi_0}(\mathbb{R}^7)$ we denote the subset of associatives inside $G_3(\mathbb{R}^7)$, the grassmannian of oriented 3-planes in \mathbb{R}^7 . It is diffeomorphic to $G_2/SO(4)$, where the action of G_2 on $\text{Im } \mathbb{O}$ restricts to the action of $SO(4)$ on $\text{Im } \mathbb{H} \oplus \mathbb{H}$ with $\text{Im } \mathbb{H}$ and \mathbb{H} isomorphic as $SO(4)$ -representations to the space of anti-self-dual forms $\Lambda_-^2 \mathbb{R}^{4*}$ and the standard vector representation \mathbb{R}^4 . As the name suggests, the associator vanishes on associative planes. In fact, associativity is tantamount to saying that the restriction to Y of the \mathbb{R}^7 valued 3-form χ_0 defined over \mathbb{R}^7 by

$$\langle \chi_0(u, v, w), x \rangle = \psi_0(u, v, w, x)$$

vanishes.

For the form χ_0 , we have the important identity

$$\chi_0(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v. \quad (6)$$

In particular, we find $u \times (u \times a) = -|u|^2 \cdot a$ if a is orthogonal to u . Further, as remarked in [3], if Y is associative, then any $u \in Y$ of norm 1 induces an hermitian structure $u \times : Y^\perp \rightarrow Y^\perp$. It follows that Y^\perp is the irreducible Clifford module of $\text{Cliff}(Y, \langle \cdot, \cdot \rangle|_Y)$. Also note that $Y^\perp \times Y^\perp \rightarrow Y$.

Finally, an oriented 4-plane X is said to be *coassociative* if and only if ψ_0 restricted to X is equal to the induced Riemannian volume form. This is equivalent to saying that the restriction of φ_0 to X vanishes. As for associatives, we find for the set of coassociatives $G^{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4)$.

2.3 G_2 manifolds

Next, consider a 7-dimensional manifold M , together with some 3-form φ . If the structure group $GL(7)$ reduces to G_2 , we say that M carries a *topological G_2 structure*. The associated G_2 principal frame bundle consists of isomorphisms between $(T_x M, \varphi_x)$ and $(\mathbb{R}^7, \varphi_0)$ for $x \in M$. In particular, formula (4) gives rise to a globally defined Riemannian metric $g = g_\varphi$ and a cross product $\times = \times_\varphi$, inducing the structure of $\text{Im } \mathbb{O}$ on any tangent space $T_x M$. Similarly, there are global counterparts $\psi = \star_\varphi \varphi \in \Omega^4(M)$ and $\chi \in \Gamma(\Lambda^3 T^* M \otimes TM)$ of ψ_0 and χ_0 .

An oriented 3-dimensional submanifold Y is called *associative* if the pull-back of φ to Y is equal to the induced Riemannian volume form. Equivalently, the pull-back of χ to Y is identically zero. An oriented 4-dimensional submanifold X is called *coassociative* if the pull-back of ψ to X is equal to the induced Riemannian volume form. Equivalently, the pull-back of φ to X is identically zero.

(Co-)Associative manifolds have the important property of being homologically volume minimising if the form φ (ψ) is closed [13]. By a result of Fernandez and Gray [10], $d\varphi = d\psi = 0$ is tantamount to saying that the holonomy group of g is contained in G_2 . Equivalently, there exist coordinates around each point such that $\varphi(x) = \varphi_0 + O(|x|^2)$.

Example: A important family of examples is provided by $K \times S^1$, where (K, ω, Ω) is a Calabi–Yau 3–fold, ω being the Kähler 2–form and Ω the holomorphic volume form. In this case, $\varphi = \operatorname{Re} \Omega + \omega \wedge dt$ and $\psi = \operatorname{Im} \Omega \wedge dt + \omega^2/2$. Further, if $C \subset M$ is a complex curve, then $Y = C \times S^1$ is associative. If $L \subset M$ is special lagrangian, then $Y = L \times \{pt\}$ is associative while $X = L \times S^1$ is coassociative.

3 The geometry of the deformation problem

Let M be a holonomy G_2 manifold and $Y \subset M$ a compact associative whose boundary ∂Y is contained in a fixed 4–submanifold $X \subset M$. We wish to study the space $\mathcal{M}_{X,Y}$ of infinitesimal deformations of Y in the class of associatives with boundary in X .

3.1 The closed case

To begin with we rederive McLean’s result for the closed case $\partial Y = \emptyset$ following Akbulut and Salur [3], who emphasise the systematic use of the cross product \times . We consider \mathcal{M}_Y , the space of infinitesimal associative deformations of Y .

An *associative deformation* of an associative Y will be a map $Y \rightarrow Y_t \subset M$, where Y_t is a family of associatives for $t \in (-\epsilon, \epsilon)$. For convenience, we denote this map also by Y_t . Since Y is compact, we may always assume, reparametrising with a time–dependent diffeomorphism if necessary, that Y_t is a *normal deformation*, i.e. the deformation vector field $\sigma_t(Y_t(p)) = \frac{\partial Y_s(p)}{\partial s}|_{s=t}$ is normal to Y_t . Now let $\nu \rightarrow Y$ denote the normal bundle over Y , equipped with the natural connection ∇ coming from the Levi–Civita connection on TM . If $\sigma \in \Gamma_Y(\nu)$ is a normal vector field on Y , then its flow ϕ_t^σ gives rise to a deformation $Y_t = \phi_t^\sigma(Y)$ of Y for t small enough. The tangent space at a point $\phi_t^\sigma(p)$ gives an element in $G_3(TM)$, the grassmannian bundle of oriented 3–planes inside TM . Taking a curve $\alpha \subset Y$ and abusing notation, we obtain a curve $\tilde{\alpha}_t(s) = \phi_t^\sigma(\alpha(s))$ in $G_3(TM)$. If Y_t is associative, then $\tilde{\alpha}_t \subset G^\varphi(TM)$, the bundle of associative planes in TM , and therefore, the derivative $\dot{\tilde{\alpha}}_t$ has to lie in $T_{\tilde{\alpha}_t} G^\varphi(TM) \subset T_{\tilde{\alpha}_t} G_3(TM)$. In the limit where $t \rightarrow 0$, we end up with the condition $\nabla_{\dot{\alpha}} \sigma \in T_\alpha G^\varphi(TM)$ inside $T_\alpha G_3(M) \cong T_\alpha^* Y \otimes \nu_\alpha$. Now Clifford multiplication \times takes the latter space to ν_α . Furthermore, we have the decomposition into $SO(4)$ irreducibles $T_\alpha^* Y \otimes \nu_\alpha = \mathfrak{so}(4)^\perp \oplus \mathbb{R}^4$ (the complement of $\mathfrak{so}(4)$ being taken in \mathfrak{g}_2) corresponding to $T_\alpha G^\varphi(TM)$ and the fibre ν_α . Appealing to Schur’s lemma, the kernel of \times is therefore $T_\alpha G^\varphi(TM)$. In other words, σ is required to lie in the kernel of the *Dirac operator*

$$\mathbf{D} : \sigma \in \Gamma_Y(\nu) \xrightarrow{\nabla} \Gamma_Y(T^*Y \otimes \nu) \xrightarrow{g} \Gamma_Y(TY \otimes \nu) \xrightarrow{\times} \Gamma_Y(\nu),$$

which written in a local orthonormal basis e_1, e_2, e_3 of Y is

$$\mathbf{D}\sigma = e_1 \times \nabla_{e_1} \sigma + e_2 \times \nabla_{e_2} \sigma + e_3 \times \nabla_{e_3} \sigma. \quad (7)$$

Summarising, we derived McLean’s

Theorem 3.1 [25] *A section $\sigma \in \Gamma_Y(\nu)$ is the deformation vector field to first order to a family of associatives, i.e. it lies in the Zariski tangent space of the infinitesimal associative deformations of Y , if and only if $\mathbf{D}\sigma = 0$.*

3.2 The geometry on the boundary

For $\partial Y \neq \emptyset$, we need to understand the geometry on the boundary of Y .

Fix a collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ of ∂Y and let u denote the inward pointing unit vector field defined on \mathcal{C} . As before, $\nu \rightarrow Y$ denotes the normal bundle as well as its restriction to ∂Y . In virtue of Section 2.2, $\nu|_{\mathcal{C}}$ carries a hermitian structure near the boundary induced by u , namely

$$G : \nu \rightarrow \nu, \quad G(x) = u \times x.$$

This acts indeed as an isometry with respect to g , as

$$g(Ga, Gb) = \varphi(u, a, u \times b) = -g(u \times (u \times b), a) = g(a, b)$$

for any $a, b \in \nu|_{\mathcal{C}}$. Let $\nu_X \subset TX|_{\partial Y}$ denote the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$.

Lemma 3.2 *For the bundle $\nu \rightarrow \partial Y$ holds the following :*

1. *The bundle ν_X is contained in ν and is stable under G .*
2. *The orthogonal complement μ_X of ν_X in ν is also stable under G .*
3. *Viewing $T\partial Y$, ν_X and μ_X as G -complex bundles, we have*

$$\bar{\mu}_X \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$$

as complex bundles, that is $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes T^{1,0}\partial Y \cong \nu_X^{1,0} \otimes \bar{K}_{\partial Y}$, where $K_{\partial Y}$ is the canonical line bundle over ∂Y .

Proof: Let us fix a local orthonormal basis (u, v, w) on the boundary by choosing locally a unit vector field $v \in T\partial Y$. We then set $w = u \times v$, which lies in $T\partial Y$ in virtue of the associativity of Y . If $a \in \nu_X$, then $g(a, u) = 0$, for $v \times w = u$ and $\varphi(v, w, a) = 0$, X being coassociative. Clearly, the vectors $a \times v$ and $a \times w$ are orthogonal to v and w as well as to u , since

$$g(a \times v, u) = \varphi(a, v, u) = -g(u \times v, a) = -g(w, a) = 0,$$

and similarly for $a \times w$. Hence $a \times v, a \times w \in \nu$. Further, these vectors are orthogonal to TX , for $a, v, w \in TX$ and X is coassociative, so that for $n \in \nu_X$ we find $g(a \times v, n) = \varphi(a, v, n) = 0$ etc. Hence $a \times v$ and $a \times w$ span μ_X . As a consequence, $u \times a \in \nu$ is orthogonal to μ_X (for $g(u \times a, a \times v) = \varphi(u, a, a \times v)$ etc.), so that ν_X is spanned by a and $u \times a = Ga$. This shows that ν_X is stable under G . On the other hand, $g(u \times (a \times v), a) = \varphi(u, (a \times v), a) = 0$ and similarly $g(u \times (a \times v), u \times a) = 0$, hence $u \times (a \times v) \in \mu_X$ which shows that μ_X is also stable under G .

The Riemann surface structure on ∂Y is induced by the hermitian structure $G = u \times$ (to keep notation tight we abuse notation and also write G for the endomorphism on $T\partial Y$ induced by $u \times$), for $Gv = u \times v = w$ and $Gw = u \times w = -v$. The map

$$a \otimes y \in \nu_X \otimes_{\mathbb{C}} T\partial Y \mapsto a \times y \in \bar{\mu}_X,$$

where we now view ν_X and $\bar{\mu}_X$ as complex line bundles via G , is well-defined and a *real* bundle isomorphism. It remains to see that it is complex-linear, i.e.

$$Ga \times y = a \times Gy = -G(a \times y).$$

This is equivalent to $(u \times a) \times y$ and $a \times (u \times y)$ being equal to $-u \times (a \times y)$. But this follows from (6) and the skew-symmetry of χ . ■

4 Ellipticity and index

Let \mathbf{D} still denote the Dirac operator (7). Since \mathbf{D} and its extension $\mathbf{D}^{\mathbb{C}} : \Gamma_Y(\nu^{\mathbb{C}}) \rightarrow \Gamma_Y(\nu^{\mathbb{C}})$ to the complexified bundle $\nu^{\mathbb{C}} = \nu \otimes \mathbb{C}$ is elliptic, we can consider its index, $\text{index}(\mathbf{D}^{\mathbb{C}}) = \dim \ker \mathbf{D}^{\mathbb{C}} - \dim \text{coker} \mathbf{D}^{\mathbb{C}}$. McLean's theorem 3.1 identifies \mathcal{M}_Y , the space of infinitesimal associative deformations, with $\ker \mathbf{D}$. We therefore refer to the index of $\mathbf{D}^{\mathbb{C}}$ as the *virtual dimension* of \mathcal{M}_Y which, Y being odd dimensional, is necessarily 0. In generic situations, one expects the cokernel of \mathbf{D} to vanish, so that by the Sard–Smale theorem \mathcal{M}_Y will be a smooth manifold of actual dimension $\dim \ker \mathbf{D}^{\mathbb{C}} = \text{index} \mathbf{D}^{\mathbb{C}} = 0$, but in non-generic situations, even for \mathcal{M}_Y smooth, the dimension might become greater.

Next let $\mathbf{B} : \Gamma_{\partial Y}(\nu) \rightarrow \Gamma_{\partial Y}(\mu_X)$ be the real operator of order 0 which projects smooth sections of $\nu = \nu_X \oplus \mu_X$ over ∂Y to μ_X . As a corollary to McLean's theorem 3.1 and Lemma 3.2, the deformation space $\mathcal{M}_{X,Y}$ can be identified with solutions of the system

$$\mathbf{D}\sigma = 0, \quad \mathbf{B}(\sigma|_{\partial Y}) = 0.$$

In view of applying the standard machinery of index theory to manifolds with boundary, we consider the extended problem over the complexified bundles $\nu_X^{\mathbb{C}}$ and $\mu_X^{\mathbb{C}}$, namely

$$\mathbf{D}^{\mathbb{C}}\sigma = 0, \quad \mathbf{B}^{\mathbb{C}}(\sigma|_{\partial Y}) = 0. \quad (8)$$

In order to compute the virtual dimension for $\mathcal{M}_{X,Y}$ we need a suitable ellipticity condition on (8).

Definition 4.1 (cf. [4]) *Let $\mathcal{D} : \Gamma_Y(S) \rightarrow \Gamma_Y(S)$ be a Dirac operator of some (complex) Clifford bundle $S \rightarrow Y$ on an odd-dimensional manifold Y with boundary. Let*

$$\mathcal{Q}_{\mathcal{D}} : \Gamma_{\partial Y}(S) \rightarrow \{\sigma|_{\partial Y} \in \Gamma_{\partial Y}(S) \mid \mathcal{D}\sigma = 0 \text{ in } Y \setminus \partial Y\}$$

*denote the associated Calderón projector [7] which projects smooth sections of $S|_{\partial Y}$ onto the space of Cauchy data of \mathcal{D} . An operator $\mathcal{B} : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V)$ of order 0 taking sections of $S|_{\partial Y}$ to sections of some complex vector bundle V , is said to define a local elliptic boundary condition if its principal symbol $\sigma(\mathcal{B})$ satisfies $\text{Im } \sigma(\mathcal{B}) \cong \pi^*V$ and $\text{Im } \sigma(\mathcal{B}) = \text{Im } (\sigma(\mathcal{B}) \circ \sigma(\mathcal{Q})) \cong \text{Im } \sigma(\mathcal{Q})$, where $\pi : T^*\partial Y \setminus 0 \rightarrow \partial Y$ denotes the natural projection.*

If \mathcal{B} defines a local elliptic boundary condition, the index

$$\text{index}(\mathcal{D}, \mathcal{B}) = \dim \ker(\mathcal{D} \oplus \mathcal{B}) - \dim \text{coker}(\mathcal{D} \oplus \mathcal{B})$$

is well-defined, finite and depends, as the usual index, only on the homotopy type of the principal symbols involved. So we may always assume, possibly after homotopically deforming the metric, that the Riemannian structure is a product on some collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ in Y . Then, if u denotes the inward pointing coordinate vector, the Dirac operator \mathcal{D} decomposes near the boundary into

$$\mathcal{D} = u \bullet (\nabla_u + \mathbf{R}),$$

where u induces a unitary automorphism $G = u \bullet$ of $S|_{\partial Y}$ squaring to minus the identity. We denote the corresponding $\pm i$ eigenspaces by S^{\pm} . The following theorem, whose proof can be also found in [4], is a valuable tool for the explicit computation of an index.

Theorem 4.1 *Let $\mathcal{D} : \Gamma_Y(S) \rightarrow \Gamma_Y(S)$ be a Dirac operator over an odd-dimensional manifold Y with boundary.*

1. *If \mathcal{P}^+ denotes the orthogonal projector¹ onto S^+ , then $(\mathcal{D}, \mathcal{P}^+)$ is a local elliptic boundary problem with vanishing index.*
2. *Let $\mathcal{B}_1 : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V_1)$ and $\mathcal{B}_2 : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V_2)$ be two orthogonal projectors onto subbundles $V_{1,2}$ of $S|_{\partial Y}$ defining a local elliptic boundary problem. Then*

$$\text{index}(\mathcal{D}, \mathcal{B}_2) - \text{index}(\mathcal{D}, \mathcal{B}_1) = \text{index}(\mathcal{B}_2 \mathcal{Q}_{\mathcal{D}} \mathcal{B}_1 : \Gamma_{\partial Y}(V_1) \rightarrow \Gamma_{\partial Y}(V_2)),$$

where on the right hand side we view \mathcal{B}_1 as an operator $\Gamma_{\partial Y}(V_1) \rightarrow \Gamma_{\partial Y}(S)$.

We are now in a position to prove the central theorem of this paper.

Theorem 4.2 *The pair $(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}})$ in (8) defines a local elliptic boundary condition with index*

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \text{index}(\bar{\partial}_{\nu_X}).$$

Remark: If the boundary is connected, then Riemann–Roch implies $\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \int_{\partial Y} c_1(\nu_X) + 1 - g$, where g is the genus of ∂Y and $c_1(\nu_X)$ is the first Chern class of ν_X with respect to the natural complex structure (or equivalently, the natural orientation) induced by u . In general, the index will therefore be

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \sum_j \int_{\Sigma_{g_j}} c_1(\nu_X|_{\Sigma_{g_j}}) + 1 - g_j$$

where Σ_{g_j} denotes a connected component of ∂Y of genus g_j .

Proof: As above, consider some collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon]$ of ∂Y on which we assume the Riemannian structure to be a product. Further, complete the inward pointing coordinate vector u to a local orthonormal basis $(v(y, t), w(y, t))$ of $T_y \partial Y \times \{t\}$ such that $u \times v = w$. Near the boundary, we have the decomposition $\mathbf{D}^{\mathbb{C}} = u \times (\nabla_u + \mathbf{R})$ with

$$\mathbf{R} = w \times \nabla_v - v \times \nabla_w, \tag{9}$$

as follows from $(a \times b) \times c = -a \times (b \times c)$ valid whenever $\{a, b, c\}$ is an orthogonal family, cf. (6). The unitary automorphism G on $\nu^{\mathbb{C}}$ with eigenspaces S^{\pm} is just $G = u \times$.

Locally, we will work with the following basis of $\nu^{\mathbb{C}}$: Choose a nowhere vanishing local section $a \in \Gamma_{\partial Y}(\nu_X)$ so that $\nu_X^{\mathbb{C}}$ is spanned by $\alpha = a - iGa$ and $\bar{\alpha} = a + iGa$ respectively, cf. Lemma 3.2. Consider then the sections $\beta = -v \times \bar{\alpha}$ and $\bar{\beta} = -v \times \alpha$. Again, the lemma implies that these span $\mu_X^{\mathbb{C}}$ locally. Further, α and β span S^+ while $\bar{\alpha}$ and $\bar{\beta}$ span S^- . As an example, take $G\alpha = Ga + ia = i\alpha$ and $G\bar{\beta} = v \times (u \times \alpha) = -i\bar{\beta}$ etc. For any subsequent matrix representation over $\nu^{\mathbb{C}}$, the ordered basis $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$ shall be used.

To show that $\mathbf{B}^{\mathbb{C}}$ defines a local elliptic boundary condition requires the principal symbol $q = \sigma(\mathbf{Q})$ of the Calderón operator $\mathbf{Q} = \mathcal{Q}_{\mathbf{D}^{\mathbb{C}}}$ associated with $\mathbf{D}^{\mathbb{C}}$. By the Calderón–Seeley theorem (as given in [4]), q is the projector onto the eigenspace of $\sigma(\mathbf{R})$ corresponding to

¹By an *orthogonal projector* we understand an operator \mathcal{P} of order 0 satisfying $\mathcal{P} = \mathcal{P}^2 = \mathcal{P}^*$.

eigenvalues with positive real part. Now with respect to our fixed local ordered basis of $\nu^{\mathbb{C}}$ around $x \in \partial Y$, v and w act via

$$v \times = \begin{pmatrix} \mathbf{0} & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & \mathbf{0} \end{pmatrix}, \quad w \times = \begin{pmatrix} \mathbf{0} & 0 & -i \\ 0 & i & 0 \\ -i & 0 & \mathbf{0} \end{pmatrix}.$$

This follows from $v \times \alpha = -\bar{\beta}$, $w \times \alpha = -u \times (v \times \alpha) = -i\bar{\beta}$ etc. For $(\eta_v, \eta_w) \in T_x^* \partial Y \setminus \{0\}$ of unit norm, we deduce from (9) (with $\eta = \eta_v + i\eta_w$) that

$$\sigma(\mathbf{R})(x, \eta) = i(\eta_v \cdot w \times - \eta_w \cdot v \times) = \begin{pmatrix} \mathbf{0} & 0 & \bar{\eta} \\ 0 & -\eta & 0 \\ \eta & 0 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & r_-(x, \eta) \\ r_+(x, \eta) & \mathbf{0} \end{pmatrix}.$$

Now $r_+(x, \eta)^* = r_-(x, \eta)$ and $r_+(x, \eta) = r_-(x, \eta)^{-1}$, so that $\sigma(\mathbf{R})(x, \eta)^* = \sigma(\mathbf{R})(x, \eta) = \sigma(\mathbf{R})(x, \eta)^{-1}$. Consequently, the eigenvalues are ± 1 , and the projector on the eigenspace associated with 1 is given by

$$q(x, \eta) = \frac{1}{2} \begin{pmatrix} \text{Id}_2 & r_-(x, \eta) \\ r_+(x, \eta) & \text{Id}_2 \end{pmatrix}.$$

On the other hand, $\mathbf{B}^{\mathbb{C}}$ is the orthogonal projector onto $\mu_X^{\mathbb{C}}$, so that its principal symbol is the matrix (taken with respect to the fixed basis of $\nu^{\mathbb{C}}$ and $\{\beta, \bar{\beta}\}$ of $\mu_X^{\mathbb{C}}$)

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is of full rank. Since

$$\sigma(\mathbf{B}^{\mathbb{C}}) \circ q(x, \eta) = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix}$$

is also of full rank, the boundary condition defined by $\mathbf{B}^{\mathbb{C}}$ is local elliptic by Definition 4.1.

It remains to compute the index. Let \mathbf{P}^+ denote the orthogonal projector onto S^+ . In virtue of Theorem 4.1 and the established local ellipticity of $\mathbf{B}^{\mathbb{C}}$,

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \text{index}(\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+ : \Gamma_{\partial Y}(S^+) \rightarrow \Gamma_{\partial Y}(\mu_X^{\mathbb{C}})).$$

But the symbol of $\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+$ is just

$$\begin{aligned} \sigma(\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+)(x, \eta) &= \sigma(\mathbf{B}^{\mathbb{C}}) \circ q \circ \sigma(\mathbf{P}^+)(x, \eta) \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} : S_x^+ \rightarrow (\mu_X^{\mathbb{C}})_x, \end{aligned}$$

where the matrix is taken with respect to the basis $\{\alpha, \beta\}$ of S^+ and $\{\beta, \bar{\beta}\}$ of $\mu_X^{\mathbb{C}}$. In particular, the symbol sends β to β and therefore acts as the identity on $\mu_X^{1,0} = S^+ \cap \mu_X^{\mathbb{C}}$.

On the other hand, the induced map $\nu_X^{1,0} = S^+ \cap \nu_X^{\mathbb{C}} \rightarrow \mu_X^{0,1} = S^- \cap \mu_X^{\mathbb{C}}$ is up to $-i$ the symbol of the Cauchy–Riemann operator $\bar{\partial}_{\nu_X}$ on $\nu_X^{1,0}$ after the identification $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes \bar{K}_{\partial Y}$ (cf. Lemma 3.2). Indeed, on a trivialisation of $\nu_X^{1,0}$ it acts as $\bar{\partial}_{\nu_X} = (\partial_1 + i\partial_2)/2$, where (x_1, x_2) are coordinates such that $\partial_1(x) = v(x)$ and $\partial_2(x) = w(x)$. Hence $\sigma(\bar{\partial}_{\nu_X})(x, \eta) = i(\eta_v + i\eta_w)/2$ which is what we wanted. \blacksquare

5 Generalisations of the deformation theorem

Boundaries in φ -free submanifolds. For the deformation problem (8) the bundle ν_X (together with the bundle μ_X it determines) is the only non-intrinsic piece of datum attached to Y , and its properties were derived using the coassociativity of X . In fact, following an idea of Harvey and Lawson, the condition on X may be relaxed as follows:

Definition 5.1 [14], [15], [16] *A 4-submanifold X of a topological G_2 manifold (M, φ) is said to be φ -free at x if $T_x X$ is φ -free, i.e. if it contains no associative 3-plane: $G_3(T_x X) \cap G^\varphi(T_x M) = \emptyset$. We call X φ -free if it is φ -free for all $x \in X$.*

Lemma 5.1 *A submanifold $X \subset M$ is φ -free at x if and only if $\varphi_{x|(T_x X)^\perp} \not\equiv 0$. In particular, the set of φ -free planes in $T_x M$ is open, and a generic 4-submanifold is φ -free.*

Proof: If $E \in G_4(T_x M)$ is not φ -free, it contains an associative 3-plane F so that E^\perp is contained in the coassociative plane F^\perp . Hence $\varphi_{x|E^\perp} \equiv 0$ which proves the sufficiency of the condition. For the converse, pick E such that $\varphi_{x|E^\perp} \equiv 0$. We need to exhibit a coassociative plane F^\perp containing E^\perp , so that E is not φ -free. Writing $E^\perp = u \wedge v \wedge w$, choose a vector a in the orthogonal complement to the linear span of $u, v, w, u \times v, u \times w, v \times w$ which is at most six dimensional. Then the 4-plane F^\perp spanned by E^\perp and a is coassociative, for $\varphi|_{F^\perp} \equiv 0$. \blacksquare

Put differently, for any $a, b \in T_x X$, we have $a \times b \notin T_x X$ and in particular, any coassociative is φ -free. From this point of view the class of φ -free submanifolds in G_2 geometry naturally matches the class of totally real submanifolds (which in particular comprises lagrangians) in Kähler geometry.

Next we wish to investigate $\mathcal{M}_{X,Y}$ under the assumption that X is φ -free. Let N_X be the orthogonal complement of $T\partial Y$ in TX , and define ν_X to be the image of N_X under the orthogonal projection $\pi : TM|_{\partial Y} \rightarrow \nu$. As before, μ_X denotes the orthogonal complement of ν_X in ν . The geometry on the boundary is specified by following

Lemma 5.2 *If X is φ -free, then*

1. *the restriction of π to N_X defines an isomorphism onto ν_X .*
2. *for any non-zero $b \in \mu_X$, $Gb = u \times b \notin \nu_X$, that is $(G\mu_X) \cap \nu_X = \{0\}$.*

Proof: The kernel of π is $TY|_{\partial Y}$. Since TX is φ -free, $N_X \cap \ker \pi = \{0\}$, whence the first assertion. Next, suppose there is an $x \in \partial Y$ and a $b_0 \in (\mu_X)_x$ of unit norm such that $Gb_0 \in (\nu_X)_x$. Let $b_1 \in (\mu_X)_x$ be a vector orthogonal to b_0 . Then Gb_1 lies in $(\nu_X)_x$, for it is orthogonal to b_1 and $g_x(b_0, Gb_1) = -g_x(Gb_0, b_1) = 0$. By the first assertion, there exist uniquely determined $n_{0,1} = m_{0,1} + k_{0,1} \in N_X$ with $\pi(n_{0,1}) = Gb_{0,1}$, where $m_{0,1} \in T_x M|_{\partial Y}$ and $k_{0,1} \in \ker \pi_x$. Since $(N_X)_x \perp T_x \partial Y$, we have in fact $n_{0,1} = Gb_{0,1} + \lambda_{0,1}u$ for $\lambda_{0,1} \in \mathbb{R}$. Then, it

is straightforward to check that the orthogonal complement of $T_x X|_{\partial Y}$ in $T_x M|_{\partial Y}$ is spanned by $A = b_0$, $B = b_1$ and $C = u - \lambda_0 Gb_0 - \lambda_1 Gb_1$. However, $v = A \times B \in T_x Y|_{\partial Y}$ belongs in fact to $T\partial Y$, for $g_x(u, v) = -g_x(Gb_1, b_0) = 0$. Consequently, $\varphi_x(A, B, C) = g_x(A \times B, C) = 0$ which contradicts the φ -freeness of TX , cf. the previous Lemma 5.1. \blacksquare

We can now extend Theorem 4.2 to this more general situation. Let Y be an associative with boundary in a φ -free X . If X intersects Y orthogonally in ∂Y , then $N_X = \nu_X$, and we can identify the space $\mathcal{M}_{X,Y}$ of infinitesimal associative deformations with boundary in X with the space of solutions of (8). For the general case, note that $N_X \oplus \mu_X$ and $\nu_X \oplus \mu_X$ are isomorphic subbundles of $TM|_{\partial Y}$, and we can homotope $\nu \rightarrow Y$ into a new bundle $N \rightarrow Y$ such that $N|_{\partial Y} = \nu_X \oplus \mu_X$. Then $\mathcal{M}_{X,Y}$ can be identified with solutions $\sigma \in \Gamma_Y(N)$ of

$$\mathbf{D}\pi(\sigma) = 0, \quad \mathbf{B}(\sigma|_{\partial Y}) = 0, \quad (10)$$

where $\pi : N \rightarrow \nu$ denotes orthogonal projection onto ν and $\mathbf{B} : \Gamma_{\partial Y}(N) \rightarrow \Gamma_{\partial Y}(\mu_X)$ orthogonal projection onto μ_X in $N|_{\partial Y}$.

Proposition 5.3 *Let $Y \subset M$ be an associative with boundary in a φ -free submanifold X . Then the virtual dimension of $\mathcal{M}_{X,Y}$ is given by $\text{index}(\bar{\partial}_{\nu_X})$.*

Proof: We restrict ourselves to the case where X intersects Y orthogonally for sake of simplicity; the general case follows from a similar argument. We choose a nowhere vanishing local section a of ν_X and extend $b = -v \times a$ to a local orthonormal trivialisation $\{b, \tilde{b}\}$ of μ_X . By the previous lemma, $G\mu_X$ can be graphed over $\nu_X \oplus \mu_X$, so that we may take \tilde{b} to be the orthogonal projection of Gb to μ_X . Let $(0, 1, s, t)$ be the coordinates of \tilde{b} with respect to the local basis $\{b, Gb, v \times b, w \times b\}$ of ν , where v is a nowhere vanishing local section of $T\partial Y$ and $w = Gv = u \times v$. With respect to the basis $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$ of $\nu^{\mathbb{C}}$ as given in the proof of Theorem 4.2, the matrix of $\sigma(\mathbf{B}^{\mathbb{C}})$ can be written as

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ z & -i & \bar{z} & i \end{pmatrix},$$

where $z = s + it$. For $\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+$, we therefore find

$$\sigma(\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+)(x, \eta) = \frac{1}{2} \begin{pmatrix} \eta & 1 \\ z + i\eta & -i - z\eta \end{pmatrix} : S_x^+ \rightarrow (\mu_X^{\mathbb{C}})_x. \quad (11)$$

The determinant of this matrix is $(-2i\eta - z - \bar{z}\eta^2)/4$, and multiplication with $\bar{\eta}$ shows this to vanish only if $\text{Re}(\bar{T}\eta) = -i$. Hence the system (10) is still elliptic. Furthermore, $F(Gb, t) = t\text{pr}_{\nu_X}(Gb) \oplus \text{pr}_{\mu_X}(Gb)$ is a global homotopy deforming $G\mu_X$ into μ_X and which in particular deforms Gb into b . Consequently, the symbol (11) is homotoped into the symbol of $\bar{\partial}_{\nu_X}$. By homotopy invariance we recover the same index as before. \blacksquare

Deformations in topological G_2 manifolds. Mutatis mutandis, Theorem 4.2 extends to topological G_2 manifolds. First, we recall the following generalisation of McLean's theorem which is due to Akbulut and Salur.

Theorem 5.4 [3] *On a topological G_2 manifold (M, φ) , the Zariski tangent space of infinitesimal associative deformations of an associative Y is given by the kernel of some twisted Dirac operator \mathbf{D}_a for which $\mathbf{D}_a - \mathbf{D}$ is an operator of order 0 that vanishes if φ is closed.*

Although the kernel will in general depend on \mathbf{D}_a , the twisting acts only as an operator of order 0 and will therefore have no effect on the symbol of \mathbf{D} . In particular, we recover the same index. Summarising, we arrive at

Theorem 5.5 *Let (M, φ) be a topological G_2 manifold and $Y \subset M$ be an associative with boundary in a φ -free submanifold X . Then the virtual dimension of $\mathcal{M}_{X,Y}$ is given by $\text{index}(\bar{\partial}_{\nu_X})$.*

6 Examples

In this section we consider examples on manifolds of holonomy G_2 .

A trivial compact example. Equip the torus $M = \mathbb{R}^7/\mathbb{Z}^7$ with its standard coordinates x^1, \dots, x^7 and the resulting flat G_2 structure. Then $Y = \{(x_1, x_2, x_3) \times \{0\} \in T^3 \times T^4 \mid 0 \leq x_1 \leq 1/2\}$ is an associative whose boundary lies in the two coassociatives $X_1 = \{0\} \times T^4$ and $X_2 = \{1/2\} \times T^4$. The induced bundles ν_{X_1} and ν_{X_2} are both trivial, so that the index vanishes.

The Calabi-Yau extension. Take a Riemann surface Σ in a Calabi-Yau 3-fold K with boundary in a fixed compact special lagrangian L . Lifting this setup to $M = K \times S^1$ as in the example of Section 2.3 gives an associative $Y = \Sigma \times S^1$ whose boundary $\partial Y = S^1 \times S^1$ is contained in the coassociative $X = L \times S^1$. Let $\mathcal{M}_{L,\Sigma}$ be the space of infinitesimal deformations of Σ through Riemann surfaces inside K with boundary in L . In [20], Katz and Liu proved that the virtual dimension of $\mathcal{M}_{L,\Sigma}$ vanishes. On the other hand, the infinitesimal deformations of $Y = \Sigma \times S^1$ through associatives boil down to deformations Σ through Riemann surfaces in K (cf. for instance [22]). Consequently, Y should be virtually rigid. Indeed, consider the orthogonal complement $\nu_L = (T\partial\Sigma)^\perp \rightarrow \partial\Sigma = S^1$ inside $TL|_{\partial\Sigma}$. By design, $\nu_{X|\partial Y} \cong p^*\nu_L$, where $p : \partial Y = S^1 \times S^1 \subset K \times S^1 \rightarrow S^1 \subset K$ is projection onto the left factor. As the bundle $\nu_L \rightarrow S^1$ admits a nowhere vanishing section being of rank 2, so does the (oriented) bundle ν_X . Consequently, $\chi(\nu_X) = c_1(\nu_X) = 0$ and we obtain the

Corollary 6.1 *Let K be a Calabi-Yau 3-fold and $\Sigma \subset K$ an embedded compact Riemann surface with connected boundary inside a compact special lagrangian L . Let $Y = \Sigma \times S^1$ in $M = K \times S^1$, and $X = L \times S^1$. Then the virtual dimension of $\mathcal{M}_{X,Y}$ vanishes.*

Coassociative germs. Take in M an associative submanifold Y with real analytic boundary ∂Y , and consider a nowhere vanishing real analytic section $a \in \Gamma_{\partial Y}(\nu)$. Since a metric of holonomy G_2 is necessarily Ricci flat, the metric is real analytic in harmonic coordinates [8], and so is the geodesic flow $\gamma_a : \partial Y \times (-\epsilon, \epsilon) \rightarrow M$ induced by a , which therefore generates an analytical submanifold N of dimension 3. Further, $\varphi(v, w, a) = 0$ for $v, w \in T\partial Y$, and since $\nabla\varphi = 0$, we conclude that the pull-back of φ to N vanishes identically. An argument of Cartan-Kähler type invoked by Harvey and Lawson [13] (see also [5]) shows that N is contained in a coassociative X whose germ around N is uniquely determined. Furthermore, ν_X is generated by a and $u \times a$, where u denotes again the inward pointing normal vector field of ∂Y . For instance, taking $M = \text{Im} \odot$ yields plenty of examples of arbitrary index.

Associative germs. In the vein of the previous example, consider a real analytic surface Σ inside a coassociative $X \subset M$. As before, Cartan-Kähler theory yields the existence of an associative Y whose germ is uniquely determined. By using a collar neighbourhood of

Σ inside Y we can construct an associative which we keep on denoting by Y for simplicity, whose boundary consists of two components $\Sigma \cup \Sigma'$. Further, we can translate X into X' containing Σ' by a suitable diffeomorphism C^1 -close enough to the identity. Of course, there is no reason for the diffeomorphism to preserve the G_2 structure, so that X' will not be coassociative. However, as φ -freeness is an open condition, X' will be still φ -free for suitable Y , and the generalised deformation theorem applies. Since Σ' is homeomorphic to Σ , but with flipped orientation, we conclude that the virtual dimension of $\mathcal{M}_{X,Y}$ vanishes.

7 A G_2 analogon of the Maslov index

Next we wish to introduce a G_2 analogon of the Maslov index, whose construction we briefly recall. Consider an almost complex manifold (M^{2m}, J) with embedded (not necessarily holomorphic) 2-disk D , whose boundary lies in a totally real oriented submanifold X^m (i.e. for any $x \in X$, $T_x X$ does not contain any J -complex line). Since D is contractible, the subbundle $TX|_{\partial D}$ may be regarded as a closed curve in the set of totally real oriented m -planes in \mathbb{C}^m after choosing some trivialisation of $TM|_D$. On the other hand, this set is parametrised by $GL_m(\mathbb{C})/GL_m(\mathbb{R})$ and is homotopy equivalent to $U(m)/O(m)$ – the set of oriented lagrangian submanifolds of \mathbb{C}^m . By the exact homotopy sequence for fibrations $\pi_1(U(m)/O(m)) \cong \mathbb{Z}$, and the *Maslov index* $\mu(\partial D)$ of ∂D is the integer corresponding to the homotopy class induced by $TX|_D$.

The natural construction in the G_2 setting should be the following. Let Y be an embedded (not necessarily associative) 3-disk inside a topological G_2 manifold M such that $\partial Y \cong S^2$ lies in some φ -free orientable submanifold X . Trivialising $TM|_Y$ and orienting X suitably yields thus a map from S^2 to the set \mathcal{P}_+ of positively oriented φ -free planes in \mathbb{R}^7 .

Proposition 7.1 *The set of positively oriented φ -free planes $\mathcal{P}_+ \subset G_4(\mathbb{R}^7)$ is homotopy equivalent to $G^{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4)$, the set of coassociatives. In particular, $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$.*

Definition 7.1 *Let Y be an embedded associative 3-disk in some topological G_2 manifold. We refer to the integer given by the natural class of $TX|_{\partial Y}$ in $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$ as the G_2 Maslov index of ∂Y , and denote it by $\mu_{G_2}(\partial Y)$.*

Proof: Instead of \mathcal{P}_+ we shall consider the dual set $\mathcal{P}_+^\perp \subset G_3(\mathbb{R}^7)$. Regard $\varphi : G_3(\mathbb{R}^7) \rightarrow \mathbb{R}$ as a smooth function on the grassmannian of oriented 3-planes. It takes values inside $[-1, 1]$, and the fibres are acted on transitively by G_2 . In particular, the two critical values ± 1 correspond to the set of associatives $G^\psi(\mathbb{R}^7)$ with $+$ or $-$ the natural orientation. Furthermore, any fibre $\varphi^{-1}(t)$ contains an element of the form

$$F_t = e_1 \wedge e_2 \wedge (te_3 + \sqrt{1-t^2}e_4)$$

with respect to a fixed G_2 frame e_1, \dots, e_7 of Section 2. To prove this, write $F_t = x \wedge y \wedge z$ for some unit vectors $x, y, z \in \mathbb{R}^7$. Now G_2 acts transitively on ordered orthonormal pairs with stabiliser $SU(2)$ [13], so that we may assume that $F_t = e_1 \wedge e_2 \wedge z$ upon transformation by a suitable element in G_2 . The $SU(2)$ action induced by the inclusion into G_2 gives rise to a decomposition $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$, where $SU(2)$ acts trivially on $\text{Im } \mathbb{H}$ and $\mathbb{H} = \mathbb{C}^2$ becomes the standard vector representation. We are still free to modify F_t without changing e_1 and e_2 by an element in $SU(2)$. Since this group acts transitively on the unit sphere in \mathbb{C}^2 , we may transform the unit vector $z = \sum_{i=3}^7 z^i e_i$ into $te_3 + \sqrt{1-t^2}e_4$ with $t = z^3 = \varphi(F_t)$. From

this one easily deduces that (a) ± 1 are the only critical points and (b) that these are non-degenerate in the sense that the Hessian of φ is non-degenerate in directions transverse to the orbits $\varphi^{-1}(\pm 1) \cong G_2/SO(4)$. Consequently, φ defines a $G = G_2$ invariant Morse function in the sense of [24]. By a theorem in the same paper, we conclude that $G_3(\mathbb{R}^7) = \varphi^{-1}([-1, 1])$ is homotopy equivalent to $\varphi^{-1}([-1, 0])$ with the disk bundle $G_2 \times_{SO(4)} D^4$ of the normal bundle over $G_2/SO(4)$ attached. But by Lemma 5.1, $\mathcal{P}_+^\perp = \varphi^{-1}([-1, 1]) - \varphi^{-1}([-1, 0])$, so that \mathcal{P}_+^\perp is homotopy equivalent to the open disk bundle of the normal bundle $G_2 \times_{SO(4)} \mathbb{R}^4 \rightarrow \varphi^{-1}(1)$. This, in turn, can be retracted to the base, which is the critical orbit $\varphi^{-1}(1) \cong G_2/SO(4)$. In particular, \mathcal{P}_+ is of the same homotopy type as $G_2/SO(4)$. Since $\pi_k(G_2) = 0$ for $k = 1, 2$ and $\pi_1(SO(4)) = \mathbb{Z}_2$, the exact homotopy sequence for fibrations yields the asserted homotopy groups. \blacksquare

A non-trivial representative of $\pi_2(G_2/SO(4))$ can be constructed as follows. Consider the embedded sphere $S^2 \hookrightarrow \text{Im } \mathbb{H} \subset \text{Im } \mathbb{H} \oplus \mathbb{H}$ with its natural complex structure $G = u \times$, where $u(x) = x$ now denotes the (outward pointing) position vector field normal to the tangent space. As a complex manifold, $S^2 \cong \mathbb{CP}^1$, and we can consider the complex rank 2 bundle $E = \mathcal{O}(2) \oplus \mathcal{O}(-1)$ (where by abuse of notation $\mathcal{O}(k)$ denotes the sheaf of sections as well as the corresponding complex line bundle of degree k). Here, $\mathcal{O}(2)$ is the tangent bundle seen as a $u \times$ -complex line bundle in $\mathbb{CP}^1 \times \text{Im } \mathbb{H}$ and $\mathcal{O}(-1)$ is the canonical bundle inside $\mathbb{CP}^1 \times \mathbb{H}$. Each fibre E_x is clearly coassociative, whence the map

$$f : x \in S^2 \mapsto E_x \in G^{\psi_0}(\mathbb{R}^7).$$

It remains to see that the resulting homotopy class $[f] \in \pi_2(G_2/SO(4))$ is non-trivial. To that end, we show its boundary $\partial[f]$ to be a generator of $\pi_1(SO(4)) \cong \mathbb{Z}_2$. By definition, a representative of $\partial[f]$ is obtained by compounding f with the collapsing map $c : (D^2, S^1) \rightarrow (S^2, N)$ (N and S being the north and south pole of S^2), lifting the resulting map $(D^2, S^1) \rightarrow (G^{\psi_0}(\mathbb{R}^7), E_N)$ to a map $F : (D^2, S^1) \rightarrow (G_2, SO(4))$ with respect to the covering map $\pi : G_2 \rightarrow G^{\psi_0}(\mathbb{R}^7)$ given by

$$(u_1, u_2, u_3) \mapsto u_1 \wedge u_2 \wedge u_3 \wedge C(u_1, u_2, u_3),$$

cf. (3) and (5), and finally restricting this lift to S^1 . So let $F : (D^2, S^1) \rightarrow (G_2, SO(4))$ be a lift. Under c the punctured disk D^\times gets mapped to $S^2 \setminus \{S\}$. There, we can construct a lift $\tilde{f} = v \wedge u \times v \wedge a : S^2 \setminus \{S\} \rightarrow G_2$ of f by taking (smooth) unit sections v and a of $\mathcal{O}(2)$ and $\mathcal{O}(-1)$. Since $\pi \circ F|_{D^\times} = \pi \circ \tilde{f} \circ c$, we have $F(x) = A_x(\tilde{f} \circ c(x))$ for $A : D^\times \rightarrow SO(4)$. In particular, $F(S^1) = A|_{S^1}(N)$, and $A|_{S^1}$ is a transition function of the bundle E . Therefore, it is of the form

$$A(t) = \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in U(2) \quad (12)$$

so that the induced homotopy class clearly generates $\pi(SO(4))$. In general, if Y is an embedded 3-disk inside some topological G_2 manifold (M, φ) with boundary in a coassociative X , then ν_X is a complex bundle of degree k , that is, $TX|_{\partial Y} = \mathcal{O}(2) \oplus \mathcal{O}(k)$. The same reasoning as before yields

$$t \in S^1 \mapsto \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{kit} \end{pmatrix} \in U(2)$$

as a representative of the homotopy class of $TX|_{\partial Y}$, which by definition is $\mu_{G_2}(\partial Y)$. Now $k = \int_{S^2} c_1(\nu_X)$ and we thus have proven

Proposition 7.2 *Let $Y \subset M$ be an embedded 3-disk with boundary in a coassociative X . If ν_X denotes again the orthogonal complement of $T\partial Y$ in TX , then*

$$\mu_{G_2}(\partial Y) = \int_{S^2} c_1(\nu_X) \bmod 2 = \text{index}(\nu_X) \bmod 2 + 1.$$

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